Generalization
# of a
Construction of Antoine

by

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GENERALIZATION OF A CONSTRUCTION OF ANTOINE

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[illegible] A. B.

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INTRODUCTION

The principal result of this thesis is the solution of the following problem, proposed by Borsuk(1):

> A étant un ensemble compact de dimension nulle situé dans l'éspace euclidien E^n de dimension $n > 3$, est-ce que le groupe fondamental de $E^n - A$ peut être différent de 0?

The problem was proposed, of course, in view of the fact that L. Antoine(2) had shown the answer to be affirmative for $n = 3$. By a generalization of Antoine's construction I shall show that the answer is affirmative for every $n \geq 3$.

The thesis is divided into three sections. Section I is devoted to the construction of the set A and to the proof that A is zero-dimensional In section II, $\pi_1(S^n - A)$ is computed explicitly and shown to be non-trivial by exhibiting a representation in the symmetric group, S_6†.

In section III it is shown that the construction cannot be accomplished in Hilbert space. More precisely, it is shown that the complement of any compact set in Hilbert space is contractible. I have been unable to determine whether a compact set of finite dimension can leave the Hilbert cube multiply connected, but have shown, however, that if there be such a set, then for some n its projection in the subspace $x_1 = x_2 = \cdots = x_n = 0$ is of infinite dimension. An analogous result holds for subsets of R^n ; namely, the projection in any (n-1)-dimensional hyperplane of a compact set in R^n, $n \geqslant 3$, whose complement is not simply connected, has positive dimension.

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† Although the fact has been generally accepted, R. P. Coelho(3) appears to have been first to publish a proof that for Antoine's set $\pi_1(S^3 - A)$ is non-trivial. This paper, as a generalization must, includes the case of Antoine's set and thus furnishes a second proof of the non-triviality of $\pi_1(S^3 - A)$.

Finally, a remark by J. W. Alexander(4) on Antoine's construction is generalized to show that for each q, $1 \leqslant q \leqslant n$, there is a q-cell in R^n, $n \geqslant 3$, whose complement is not simply connected.

SECTION I

CONSTRUCTION OF THE SET A

First we give a brief sketch of the construction. The set A will consist of the intersection of a decreasing sequence of compact sets,

$A_0 \supset A_1 \supset A_2 \supset \cdots$,

where

A_0 is a set in Euclidean space R^n formed by rotating a circle with its interior about n-2 hyperplanes of dimension n-2. The hyperplanes will be so chosen that A_0 is homeomorphic to the topological product of a 2-cell with n-2 circles. We shall hereafter refer to such a set as an n-tube.

A_1 consists of k disjoint n-tubes , $T_1, \cdots, T_k$, which are linked and embedded in A_0 in such a manner that the injection $\pi_1(R^n - A_0) \to \pi_1(R^n - A_1)$ is an isomorphism into.

A_i consists of k^i disjoint n-tubes situated so that each n-tube of A_{i-1} contains k n-tubes of A_i embedded in a manner homeomorphic to the embedding of A_1 in A_0.

The successive embeddings $A_i \subset A_{i-1}$, however, are accomplished so that the diameters of the n-tubes comprising A_i tend to zero as $i \to \infty$. Thus it is evident that A is compact and zero-dimensional. We next adjoin the point at infinity and consider A, A_0, A_1, $\cdots$ as lying in S^n. To prove that $\pi_1(S^n - A)$ is non-trivial, we shall compute it as the limit of the direct sequence, $\{\pi_1(S^n - A_i)\}$, and exhibit a non-trivial representation of it in the symmetric group S_6.

To prove that A is zero-dimensional and to compute its group, we need the following lemma:

Lemma IA. Let d_r, d_s , D_s be arbitrary real numbers with $0<d_s\leq D_s$ Let S be a compact set in R^n contained in the set defined by $x_r = d_r$ and $0<d_s\leq x_s\leq D_s$. Let $\widetilde{S}$ be the set generated by rotating S about the hyperplane defined by $x_r = d_r$, $x_s = 0$, or more explicitly,

$$\widetilde{S} = \{x \mid x \in R^n + \exists y \in S + \exists \theta \ni: x_i = y_i \text{ if } i \neq r \text{ or } s \text{ and } x_r = y_s \sin\theta \text{ , } x_s = y_s \cos\theta \}$$

Then

a) for each $(y, \theta) \in S \times$ Real numbers mod 2π , the correspondence, $(y,\theta) \rightarrow x$, where $x_i = y_i$, $i \neq r$ or s , $x_r = y_s \sin\theta$, $x_s = y_s \cos\theta$, is a homeomorphism onto $\widetilde{S}$. We can therefore use the pair (y,θ) as a set of coordinates for $\widetilde{S}$.

b) if $\widetilde{U}$ is the set in $\widetilde{S}$ consisting of all points with representations (u,θ) for which

$$u \in U \subset S \text{ , } \alpha \leq \theta \leq \beta \text{ , } \beta - \alpha \leq 2\pi$$

then

$$\max \{\text{diam } U, d_s\, p(\beta-\alpha)\} \leq \text{diam } \widetilde{U} \leq \text{diam } U + D_s\, p(\beta-\alpha) \quad ,$$

where

$$p(\theta) = \begin{cases} 2\sin\frac{1}{2}\theta & \text{if } 0 \leq \theta \leq \pi \\ 2 & \text{if } \pi \leq \theta \leq 2\pi \text{ .} \end{cases}$$

c) if $x \in \widetilde{S}$ then $d_r - D_s \leq x_r \leq d_r + D_s$!

Proof:

a) Is easily verified by analytical geometry.

b) Let u be a point in U. Then the points (u,α) and (u,β) are in $\overset{\frown}{U}$, hence

$$\rho((u,\alpha),(u,\beta)) = \sqrt{u_s^2(\sin\beta-\sin\alpha)^2 + u_s^2(\cos\beta-\cos\alpha)^2}$$

$$= 2u_s \sin\tfrac{1}{2}(\beta-\alpha) \geqslant 2d_s \sin\tfrac{1}{2}(\beta-\alpha)$$

If, moreover, $\beta-\alpha \geqslant \pi$, then $(u,\alpha+\pi)$ is also in U and $\rho((u,\alpha),(u,\alpha+\pi)) \geqslant 2d_s \sin \pi/2 = 2d_s$. Hence diam $\overset{\frown}{U} \geqslant d_s\, p(\beta-\alpha)$. Since $\overset{\frown}{U}$ contains an isometric copy of U, we also have diam $\overset{\frown}{U} \geqslant$ diam U. Hence, combining the above

$$\text{diam}\, \overset{\frown}{U} \geqslant \max\left\{\text{diam}\, U,\ d_s\, p(\beta-\alpha)\right\}.$$

Now let (u,θ) and $(\bar{u},\bar{\theta})$ be any two points of $\overset{\frown}{U}$. Then

$$\rho((u,\theta),(\bar{u},\bar{\theta})) \leqslant \rho((u,\theta),(u,\bar{\theta})) + \rho((u,\bar{\theta}),(\bar{u},\bar{\theta}))$$

$$\leqslant 2u_s \sin\tfrac{1}{2}|\theta-\bar{\theta}| + \rho(u,\bar{u})$$

$$\leqslant D_s\, p(\beta-\alpha) + \text{diam}\, U .$$

c) Consider a point $x \in \overset{\frown}{S}$. Then there is a point $y \in S$ such that $(x_r-d_r)^2 + x_s^2 = y_s^2$. Hence x_r-d_r is maximum or minimum when $x_s = 0$ and y_s assumes its maximum value. That is, $|x_r-d_r| \leqslant D_s$.

From this we see that $\max\limits_{x\in\overset{\frown}{S}} x_r \leqslant d_r + D_s$

and

$$\min_{x\in\overset{\frown}{S}} x_r \geqslant d_r - D_s .$$

Construction of A_0 .

Choose a set of positive numbers $r_1, r_2, \cdots, r_n$ such that

$$\text{I.}\qquad \begin{cases} d_2 = r_2 - r_1 > 0 & , \quad D_2 = r_2 + r_1 \\ d_3 = r_3 - r_1 > 0 & , \quad D_3 = r_3 + r_1 \\ d_4 = r_4 - D_3 > 0 & , \quad D_4 = r_4 + D_3 \\ \quad\vdots & \qquad\quad \vdots \\ d_{n-1} = r_{n-1} - D_{n-2} > 0 & , \quad D_{n-1} = r_{n-1} + D_{n-2} \\ r_n = 0 \end{cases}$$

Let $t^{(2)}$ be the circular 2-cell in R^n defined by

$$\begin{gathered} x_1 = 0 \\ (x_2 - r_2)^2 + (x_3 - r_3)^2 \leq r_1^2 \\ x_4 = r_4 \\ x_5 = r_5 \\ \vdots \\ x_n = r_n \quad , \end{gathered}$$

Then if $x \varepsilon t^{(2)}$ we have

$$0 < d_2 = r_2 - r_1 \leq x_2 \leq r_2 + r_1 = D_2$$

So $t^{(2)}$ satisfies the conditions of lemma A with respect to the planes $x_1 = x_2 = 0$ $(r=1, s=2)$. Hence if we define $t^{(3)}$ to be the set generated by rotating $t^{(2)}$ about $x_1 = x_2 = 0$, the following statements hold as a result of lemma A:

$a_{(3)}$: $t^{(3)}$ is a 3-tube, and $t^{(3)}$ coordinates, (u, θ_2) , $u \varepsilon t^{(2)}$, $\theta_2 \varepsilon$ real numbers mod 2π , can be set up in $t^{(3)}$.

$\underline{b_{(3)}}$: For any set $U^{(3)}$

$$U^{(3)} = \{(u,\theta_2) \mid u \varepsilon \mathcal{U}^{(2)} \subset t^{(2)}, \alpha_2 \leqslant \theta_2 \leqslant \beta_2\},$$

we have

$$\max\left(\text{diam } U^{(2)}, d_2\, \rho(\gamma_2)\right) \leqslant \text{diam } \mathcal{U}^{(3)} \leqslant \text{diam } \mathcal{U}^{(2)} + D_2\, \rho(\gamma_2),$$

where $\gamma_2 = \beta_2 - \alpha_2$.

$\underline{c_{(3)}}$: $x \varepsilon t^{(3)} \Rightarrow d_3 \leqslant x_3 \leqslant D_3$ and $x_j = r_j$, $4 \leqslant j \leqslant n$.

We now define inductively for $4 \leqslant i \leqslant n$,

$t^{(i)} =$ the set generated by rotating $t^{(i-1)}$ about the plane $x_{i-1} = 0$, $x_i = r_i$.

We now claim that the following propositions hold, for $3 \leqslant i \leqslant n$;

$\underline{\underline{a_{(i)}}}$: $t^{(i)}$ is an i-tube, and we may set up $t^{(i)}$-coordinates, $(u, \theta_2, \theta_3, \cdots, \theta_{i-1})$ with $u \varepsilon t^{(2)}$, and $\theta_j \varepsilon$ reals mod 2π , where θ_{i-1} is the angle through which $(u, \theta_2, \theta_3, \cdots \theta_{i-2})$ is rotated to obtain the point in question.

$\underline{\underline{b_{(i)}}}$: For any set

$$U^{(i)} = \{(u, \theta_2, \cdots, \theta_{i-1}) \mid u \varepsilon \mathcal{U}^{(2)} \subset t^{(2)}, \alpha_j \leqslant \theta_j \leqslant \beta_j, j = 2, \cdots, i-1\}$$

we have

$$\max_{j=2,\cdots i-1}\left(\text{diam } U^{(2)}, d_j\, \rho(\gamma_j)\right) \leqslant \text{diam } U^{(i)} \leqslant \text{diam } U^{(2)} + \sum_{j=2}^{i-1} D_j\, \rho(\gamma_j),$$

where

$$\gamma_j = \beta_j - \alpha_j$$

$\underline{\underline{c_{(i)}}}$: $x \varepsilon t^{(i)} \Rightarrow d_i \leqslant x_i \leqslant D_i$ and $x_j = r_j$ for $i+1 \leqslant j \leqslant n$.

We have already indicated that $a_{(3)}$, $b_{(3)}$, and $c_{(3)}$ hold, so we need only show that $a_{(i-1)}$, $b_{(i-1)}$, and $c_{(i-1)}$ imply $a_{(i)}$, $b_{(i)}$, and $c_{(i)}$. Now $c_{(i-1)}$ states that

$$0 < d_{i-1} \leq x_{i-1} \leq D_{i-1} \text{ and } x_i = r_i \text{ for } x \in t^{(i-1)} .$$

So taking $S = t^{(i-1)}$, $r = i$, $s = i-1$, we see that the conditions of lemma A are fulfilled and that $t^{(i)}$ corresponds to $\overleftarrow{S}$. Hence,

$a_{(i)}$ follows immediately from a and $a_{(i-1)}$ when we replace

$((u, \theta_2, \ldots, \theta_{i-2}), \theta_{i-1})$ by $(u, \theta_2, \ldots \theta_{i-2}, \theta_{i-1})$.

$b_{(i)}$ follows when we apply b to obtain

$$\max\left(\operatorname{diam} U^{(i-1)}, d_{i-1}\,\rho(\gamma_{i-1})\right) \leq \operatorname{diam} U^{(i)} \leq \operatorname{diam} U^{(i-1)} + D_{i-1}\,\rho(\gamma_{i-1}) ,$$

and then apply the limits on diam $U^{(i-1)}$ obtained from $b_{(i-1)}$.

$c_{(i)}$ follows from c, $c_{(i-1)}$, and the fact that the higher coordinates are unchanged during the rotation which generates $t^{(i)}$.

We have thus established $a_{(n)}$, $b_{(n)}$ and $c_{(n)}$. We take

$$A_0 = t^{(n)} = T.$$

For convenience of description, the coordinates described in $a_{(n)}$ will be called T-coordinates, and a set of the typed described in $b_{(n)}$, will be called a T-box, and partially described by

$$U \equiv \left\{U^{(2)}, \gamma_2, \gamma_3, \ldots \gamma_{n-1}\right\} .$$

Construction of A_1.

Let $t_1^{(3)}$, $t_2^{(3)}$, $\cdot\cdot t_k^{(3)}$ be a chain of cyclically linked disjoint 3-tubes contained in $t^{(3)}$ and looping once around the axis of $t^{(3)}$. We may assume that they are all similar and that k is large enough so that $t_i^{(3)}$ is generated by a circular 2-cell $t_i^{(2)}$ of radius ρ_1, rotated about an axis ℓ_i at a distance ρ_2 from the center of $t_i^{(2)}$. Let $T_1, T_2, \ldots T_k$ be the n-tubes generated by $t_1^{(3)} \ldots t_k^{(3)}$, respectively, as they are carried along with $t^{(3)}$ during the successive rotations required to produce $T = t^{(n)}$. The set A_1 we define by

$$A_1 = \bigcup_{i=1}^{k} T_i \quad .$$

Construction of $A_2, A_3, \ldots \ldots$

Since each of the sets T_i is an n-tube constructed in a manner similar to T, we see that we can introduce T_i-coordinates in T_i which we denote by

$$(v, \varphi_2, \varphi_3, \cdots, \varphi_{n-1})_i \quad , v \in t_i^{(2)}$$

Moreover, we note that if $x \in T_i$, then it has T-coordinates: $(u, \theta_2, \theta_3, \cdots, \theta_{n-1})$, and T_i - coordinates: $(v, \varphi_2, \varphi_3, \cdots, \varphi_{n-1})_i$,
and that

$$\underline{(1)} \qquad \theta_3 = \varphi_3, \theta_4 = \varphi_4, \cdots, \theta_{n-1} = \varphi_{n-1},$$

since the rotations generating T and T_i coincide after the first one.

Now let g_i be a linear homeomorphism of the 2-cell $t^{(2)}$ onto the 2-cell $t_i^{(2)}$, and define the homeomorphisms

$$f_i : T \cong T_i$$

by

$$\underline{(2)} \qquad f_i(u, \theta_2, \theta_3 \cdots, \theta_{n-1}) = (g_i(u), \theta_3, \theta_4, \cdots, \theta_{n-1}, \theta_2)_i \quad .$$

Thus f_i is what might be called the natural mapping of T onto T_i, except that the angular coordinates are permuted.

We now define

$$T_{ij} = f_i(T_j) = f_i f_j(T)$$

and

$$A_2 = \bigcup_{i,j=1}^{k} T_{ij}$$

Let α be an array of integers, each between 1 and k,

$$\alpha = i_1 i_2 \cdots i_\ell \quad , \quad 1 \le i_j \le k .$$

ℓ will be called the length of α or $\ell(\alpha)$. By the notation αi we shall understand the array $i_1 \cdots i_\ell i$.

For such an array α , we define

$$f_\alpha = f_{i_1} f_{i_2} \cdots f_{i_\ell} ,$$

and

$$T_\alpha = f_\alpha(T) .$$

We now define the set A_ℓ by

$$A_\ell = \bigcup_{\ell(\alpha)=\ell} T_\alpha .$$

Thus it is apparent that $A_{\ell+1}$ consists of $k^{\ell+1}$ disjoint n-tubes such that each n-tube T_α of A_ℓ contains k n-tubes of $A_{\ell+1}$, $T_{\alpha 1}$, $T_{\alpha 2}$, ... $T_{\alpha k}$, embedded homeomorphically to the embedding of A_1 in A_0. Namely,

$$(T_\alpha; T_{\alpha 1}, \cdots, T_{\alpha k}) = f_\alpha(T; T_1, \cdots, T_k) .$$

Thus A_0, A_1, . . . is a monotone decreasing sequence of compact sets and hence

$$A = \bigcap_{i=0}^{\infty} A_i \quad \text{is compact.}$$

Zero-Dimensionality of A.

We note that for every positive integer ℓ, each point of A is contained in the interior of some T_α for which $\ell(\alpha) = \ell$, and that $A \cap \operatorname{bdy} T_\alpha = 0$. Hence if we can prove that diam $T_\alpha \to 0$ as $\ell(\alpha) \to \infty$, it will follow that A is zero-dimensional. This will be apparent as soon as we have established the following theorem:

Theorem IA. It is possible to choose $r_1, r_2, \cdots r_n$, ρ_1 and ρ_2 in accordance with the construction and such that for every T-box, U, there exist T-boxes $U_1, \ldots . U_k$ such that $f_i(U) \subset U_i$, $i = 1, \cdots, k$ and diam $U_i \leqslant \frac{n-1}{n}$ diam U.

To prove this theorem we first establish

Lemma IB. Let $U_1^{(3)}$ be a $t_1^{(3)}$- box

$$U_1^{(3)} \cong \{U_1^{(2)}, r_2\}_1 .$$

Then $U_1^{(3)}$ is contained in a $t^{(3)}$ box

$$U^{(3)} \cong \{U^{(2)}; \overline{r_2}\};$$

such that

$$\operatorname{diam} U^{(2)} \leqslant \operatorname{diam} U_1^{(2)} + (\rho_1 + \rho_2)\rho(r_2)$$

and

$$d_2 \rho(\overline{r}_2) \leqslant \operatorname{diam} U_1^{(2)} + (\rho_1 + \rho_2)\rho(r_2)$$

Proof: The set $U^{(2)}$ is obtained by "projecting" $U_1^{(3)}$ back into $t^{(2)}$, the generating 2-cell of $t^{(3)}$. The first inequality simply states that the diameter of the "projection" is equal to or less than that of $U_1^{(3)}$.

If $[\alpha, \beta]$ is the smallest angular segment of $t^{(3)}$ containing $U_1^{(3)}$, and $\overline{r}_2 = \beta - \alpha$, then the second inequality states that the diameter of the smallest circular arc subtended by the interval $[\alpha, \beta]$ is less than the diameter of $U_1^{(3)}$.

I take these statements to be sufficiently evident to justify omission of a more formal proof, and include figure 1 as a visual aid.

Proof of Theorem IA.

It is obviously sufficient to prove the theorem for i = 1.

Let

$$U \cong \{U^{(2)}, r_2, r_3, \ldots, r_{n-1}\}$$

Then, referring to (1), we see that $f_1(U)$ is a T_1-box

$$f_1(U) \cong \{g_1(U^{(2)}), r_3, r_4, \ldots, r_{n-1}, r_2\}_1$$

Recalling that by (2), the T- and T_1 - coordinates of a point agree except for the first two, we have upon applying lemma B,

$$f_1(U) \subset U_1, \text{ a T-interval,}$$

$$U_1 \cong \{U_1^{(2)}, \overline{r}_3, r_4, \ldots, r_{n-1}, r_2\}$$

such that

$$\operatorname{diam} U_1^{(2)} \leqslant \operatorname{diam} g_1(U^{(2)}) + (\rho_1 + \rho_2)\, p(r_3)$$

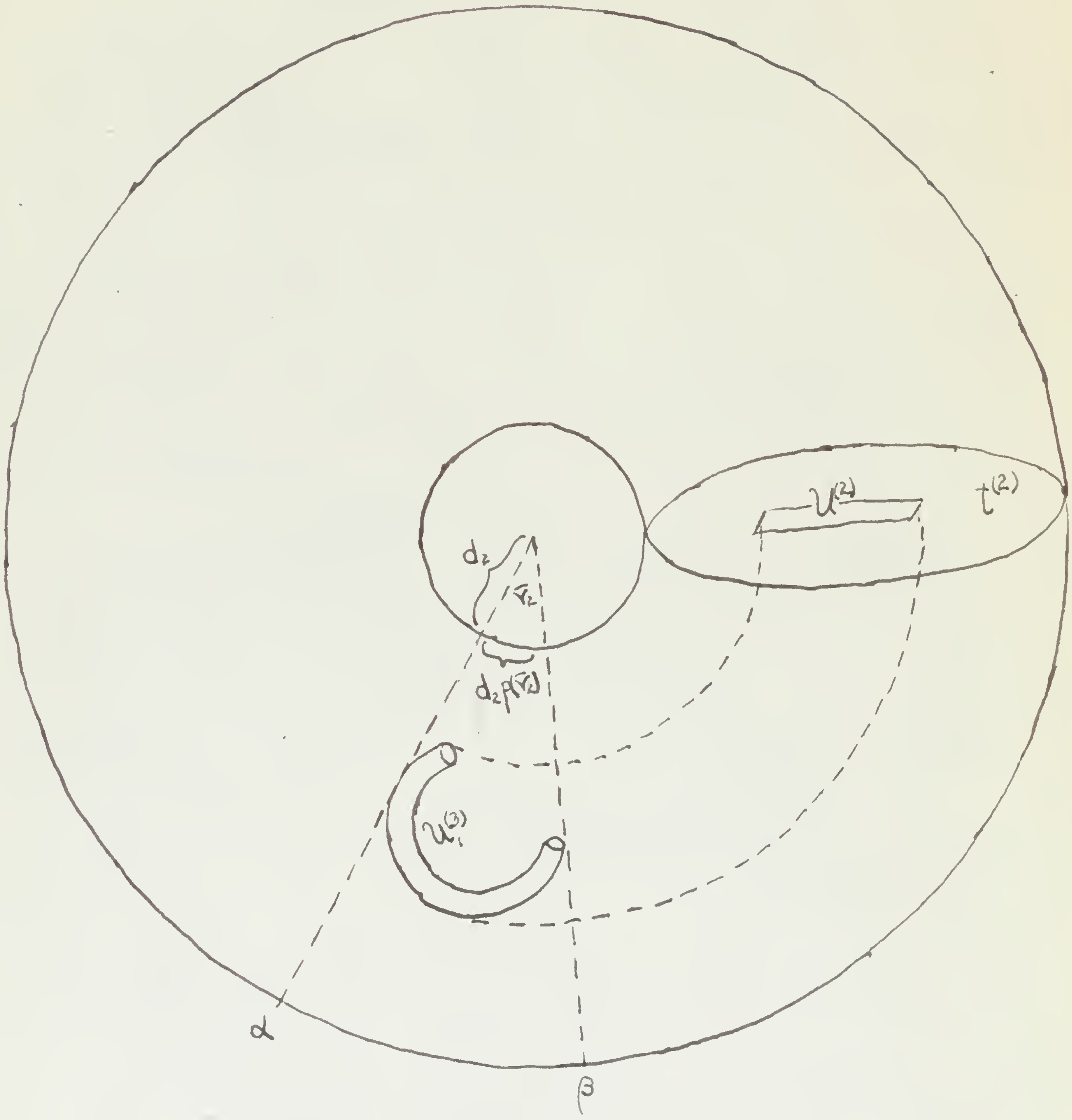
and

$$d_2\, p(\overline{r}_3) \leqslant \operatorname{diam} g_1(U^{(2)}) + (\rho_1 + \rho_2)\, p(r_3)$$

Hence, combining this with proposition $b_{(n)}$, we see that

$$\begin{aligned} \operatorname{diam} U_1 &\leqslant \operatorname{diam} U_1^{(2)} + D_2\, p(\overline{r}_3) + D_3\, p(r_4) + \cdots + D_{n-1}\, p(r_2) \\ &\leqslant \left(1 + \frac{D_2}{d_2}\right)\left(\operatorname{diam} g_1(U^{(2)}) + (\rho_1 + \rho_2)\, p(r_3)\right) + D_3\, p(r_4) + \cdots + D_{n-1}\, p(r_2) \end{aligned}$$

Now since g_1 is a 1-1 linear mapping of $t^{(2)}$ onto $t_1^{(2)}$, we know figure 1.



that the set $U^{(2)}$ is reduced in diameter by the ratio $\rho_1 : r_1$.

Hence

$$\operatorname{diam} U_1 \leqslant \left(1+\frac{D_2}{d_2}\right)\left(\frac{\rho_1}{r_1}\operatorname{diam} U^{(2)} + \frac{\rho_1+\rho_2}{d_3} d_3 p(r_3)\right) + \frac{D_3}{d_4} d_4 p(r_4) + \cdots + \frac{D_{n-1}}{d_2} d_2 p(r_2)$$

Suppose that we can choose

II. $1+\frac{D_2}{d_2} < 3$, $\rho_1/r_1 < \frac{1}{3n}$, $\frac{\rho_1+\rho_2}{d_3} < \frac{1}{3n}$, $\frac{D_3}{d_4} < \frac{1}{n}$; $\cdots$ $\frac{D_{n-2}}{d_{n-1}} < \frac{1}{n}$, $\frac{D_{n-1}}{d_2} < \frac{1}{n}$.

In this case we would have

$$\begin{aligned} \operatorname{diam} U_1 &\leqslant \frac{1}{n}\left(\operatorname{diam} U^{(2)} + d_2 p(r_2) + \cdots\cdots + d_{n-1} p(r_{n-1})\right) \\ &\leqslant \frac{n-1}{n} \max_i \left(\operatorname{diam} U^{(2)}, d_i p(r_i)\right) \leqslant \frac{n-1}{n} \operatorname{diam} U \end{aligned}$$

Hence to complete the proof of the theorem we need to show that our quantities can be chosen subject to the inequalities I (p.6), and II. These choices can be carried out in the following order:

Choose ρ_1 arbitrarily.

Choose $\rho_2 > \rho_1$ and such that the linkages are possible.

Choose $r_1 > 3n\rho_1$ and such that $t_i^{(3)}$ fits inside $t^{(3)}$.

Choose $r_3 > r_1 + 3n(\rho_1+\rho_2)$. This implies $\frac{\rho_1+\rho_2}{d_3} < \frac{1}{3n}$

Choose $r_4 > (n+1) D_3$. This implies $\frac{D_3}{d_4} < \frac{1}{n} < 1$

Choose $r_5 > (n+1) D_4$. This implies $\frac{D_4}{d_5} < \frac{1}{n} < 1$

$\vdots$

Choose $r_{n-1} > (n+1) D_{n-2}$. This implies $\frac{D_{n-2}}{d_{n-1}} < \frac{1}{n} < 1$

Now since $D_3, D_4, \cdots, D_{n-1}$, do not involve r_2 , all of the choices so far have been independent of r_2 . Hence we may choose

$$r_2 > r_1 + n D_{n-1} > 3 r_1 > r_1$$

This implies that

$$\left(1 + \frac{D_2}{d_2}\right) < 3 \qquad ,$$

$$\frac{D_{n-1}}{d_2} < \frac{1}{n} \qquad ,$$

and $$r_2 > r_1 \qquad .$$

Thus all the inequalities of I and II are satisfied in a manner consistent with the construction.

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SECTION II

COMPUTATION OF $\pi_1(S^n - A)$

To make the computation easier to follow, we first treat the case $n = 4$, and later generalize.

We have, then, that $T = t^{(4)} = A_0$ consists of the 3-tube indicated in the accompanying figure, rotated about the (x_1, x_2) - plane.

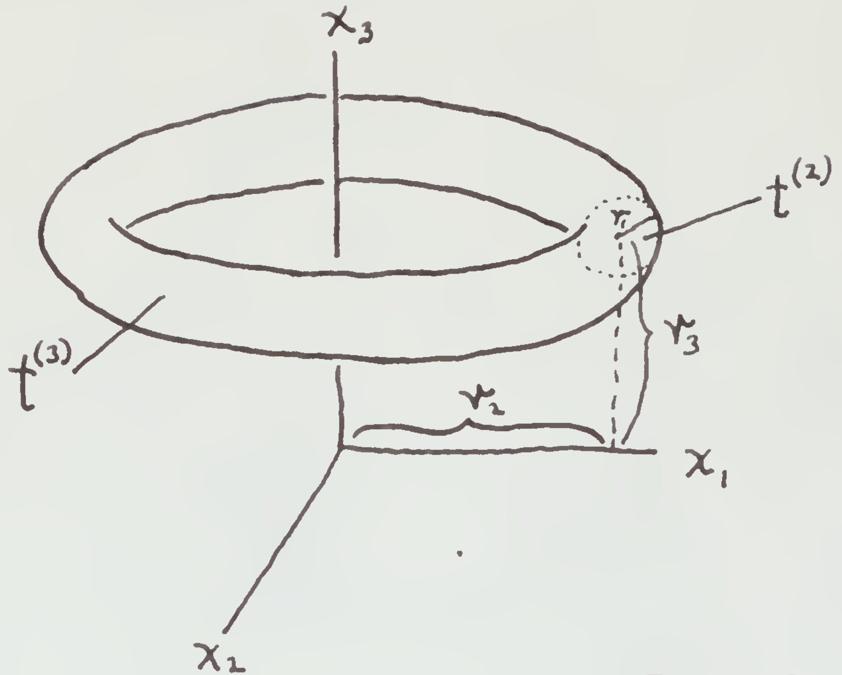


Figure 2.

We first compute $\pi_1(t^{(3)} - \bigcup_1^k t_i^{(3)})$. Denote

$$\sigma = \bigcup_1^k t_i^{(3)} \quad , \quad t' = S^3 - t^{(3)} .$$

We have then

$$t^{(3)} - \sigma = S^3 - (S^3 - (t^{(3)} - \sigma)) = S^3 - t' \cup \sigma .$$

Now t' is a 3-tube linking σ. To compute $\pi_1(S^3 - t' \cup \sigma)$ we may retract t' and $t_i^{(3)}$ upon their lines of centers, thus obtaining a linear graph. We assume that k is even and that the resulting linear graph has the following projection:

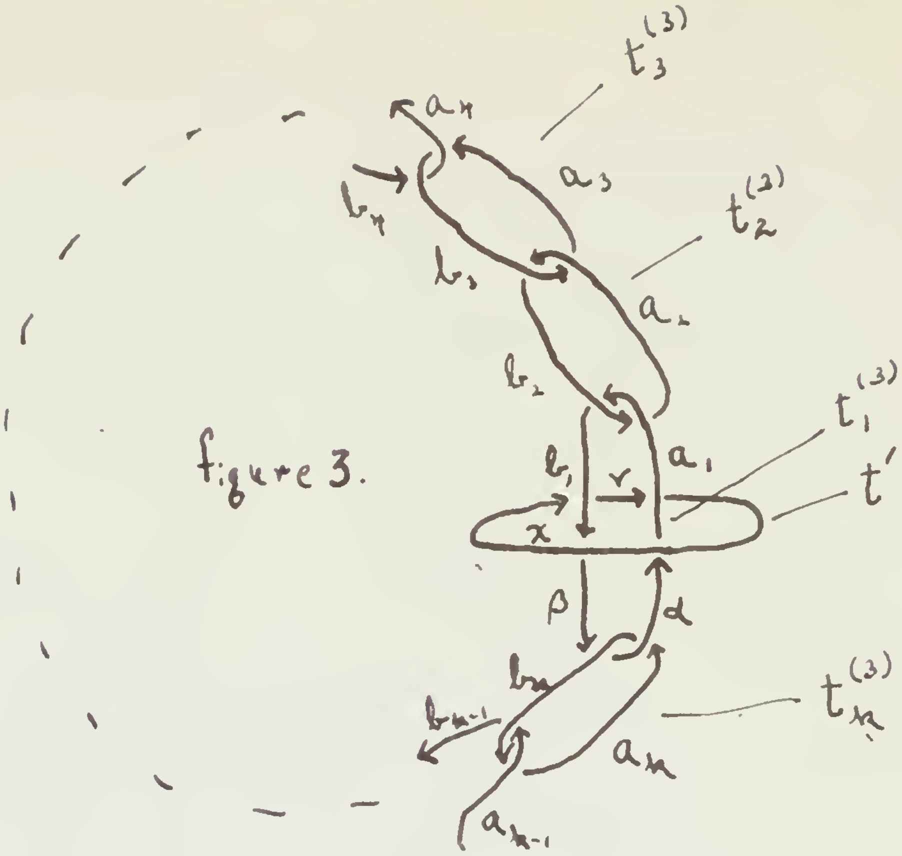


figure 3.

We further assume that the orientations indicated in the figure are in accord with the angular coordinate system, i.e., the point $(u, \theta_2)_i$ in $t_i^{(3)}$ moves in the direction of the arrow as θ_2 increases. Labelling the edges of the linear graph as indicated, we find by a standard procedure that $\pi_1(S^3 - t' \cup \sigma)$ is given by

Generators: $x, a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n, \alpha, \beta, \gamma$.

and

Relations: (Using the usual notation: $uvu^{-1}=v^{u}$, $uv^{-1}u^{-1}=v^{-u}$)

(1) $\alpha = a_1^{x}$

(2) $\gamma = x^{a_1}$

(3) $\gamma = x^{b_1}$

(4) $\beta = b_1^{x}$

(a_1) $a_1 = b_1^{b_2}$

(b_2) $a_2^{a_1} = b_2 = a_2^{a_3}$

(a_3) $b_3^{b_2} = a_3 = b_3^{b_4}$

(b_4) $a_4^{a_3} = b_4 = a_4^{a_5}$

$\vdots \qquad \vdots \qquad \vdots$

(a_{k-1}) $b_{k-1}^{b_{k-2}} = a_{k-1} = b_{k-1}^{b_k}$

(b_k) $a_k^{a_{k-1}} = b_k = a_k^{\alpha}$

(α) $\alpha = \beta^{b_k}$

If we set $y = a_1^{-1} b_1$, then from (b_2) we obtain

$$y = a_2^{-1} a_1^{-1} b_2 b_1$$

and from (a_1),

$$a_2^{-1} a_1^{-1} b_2 b_1 = a_2^{-1} b_2 \quad .$$

From the succeeding relations, we likewise obtain

$$y = a_1^{-1} b_1 = a_2^{-1} b_2 = \cdots\cdots = a_k^{-1} b_k = \alpha^{-1}\beta \quad .$$

Thus if we introduce y we can eliminate $b_1, \ldots . b_k$, from the set of generators. Also, relations (1) through (4) permit us to eliminate α, β, γ. To determine the relations for the for the reduced set of generators, we first equate (2) and (3), obtaining

$$a_1 x a_1^{-1} = b_1 x b_1^{-1},$$

or $$[x, y] = 1 \quad .$$

The relations (b_2) yield

$$a_1 a_2 a_1^{-1} = a_3 a_2 a_3^{-1} = b_2$$

or $$[a_2^{-1}, a_1] = [a_2^{-1}, a_3] = y$$

Similarly, (b_4), (b_6), ... (b_{k-2}) yield respectively,

$$\begin{aligned} y &= [a_4^{-1}, a_3] = [a_4^{-1}, a_5] \\ y &= [a_6^{-1}, a_5] = [a_6^{-1}, a_7] \\ &\vdots \qquad\qquad \vdots \\ y &= [a_k^{-1}, a_{k-1}] = [a_k^{-1}, a_1^x], \end{aligned}$$

this last relation resulting from replacing α by its value given in (1).

Thus we find that $\pi_1(t^{(3)} - \sigma)$ is given by

Generators: $x, y, a_1, a_2, \dots a_k$

Relations:

$$\begin{aligned} [x, y] &= 1 \\ y &= [a_2^{-1}, a_1] = [a_2^{-1}, a_3] \\ &= [a_4^{-1}, a_3] = [a_4^{-1}, a_5] \\ &\vdots \qquad\qquad \vdots \\ &= [a_k^{-1}, a_{k-1}] = [a_k^{-1}, a_1^x] \end{aligned}$$

That these constitute a complete set of relations is easily checked by substituting $a_i y$ for b_i and αy for β in the original relations, and noting that the resulting relations are consequences of those shown on the preceding page. For example (a_3) becomes

$$a_2 y a_3 y y^{-1} a_2^{-1} = a_3 = a_4 y a_3 y y^{-1} a_4^{-1}$$

or

$$y = [a_2^{-1}, a_3] = [a_4^{-1}, a_3] \quad .$$

We must now consider the geometrical representations of the elements x, y, a_i. Since $\pi_1(t^{(3)} - \sigma) \simeq \pi_1(\overline{t^{(3)} - \sigma})$ we will hereafter treat the latter group.

Choose the point $p \varepsilon t^{(2)}$, whose cartesian coordinates are $(0, r_2 + r_1, r_2, 0)$, as base point for the fundamental group. Assume that the $t_i^{(2)}$ were so chosen that $p_i = f_i(p)$ are located as indicated in figure 4.

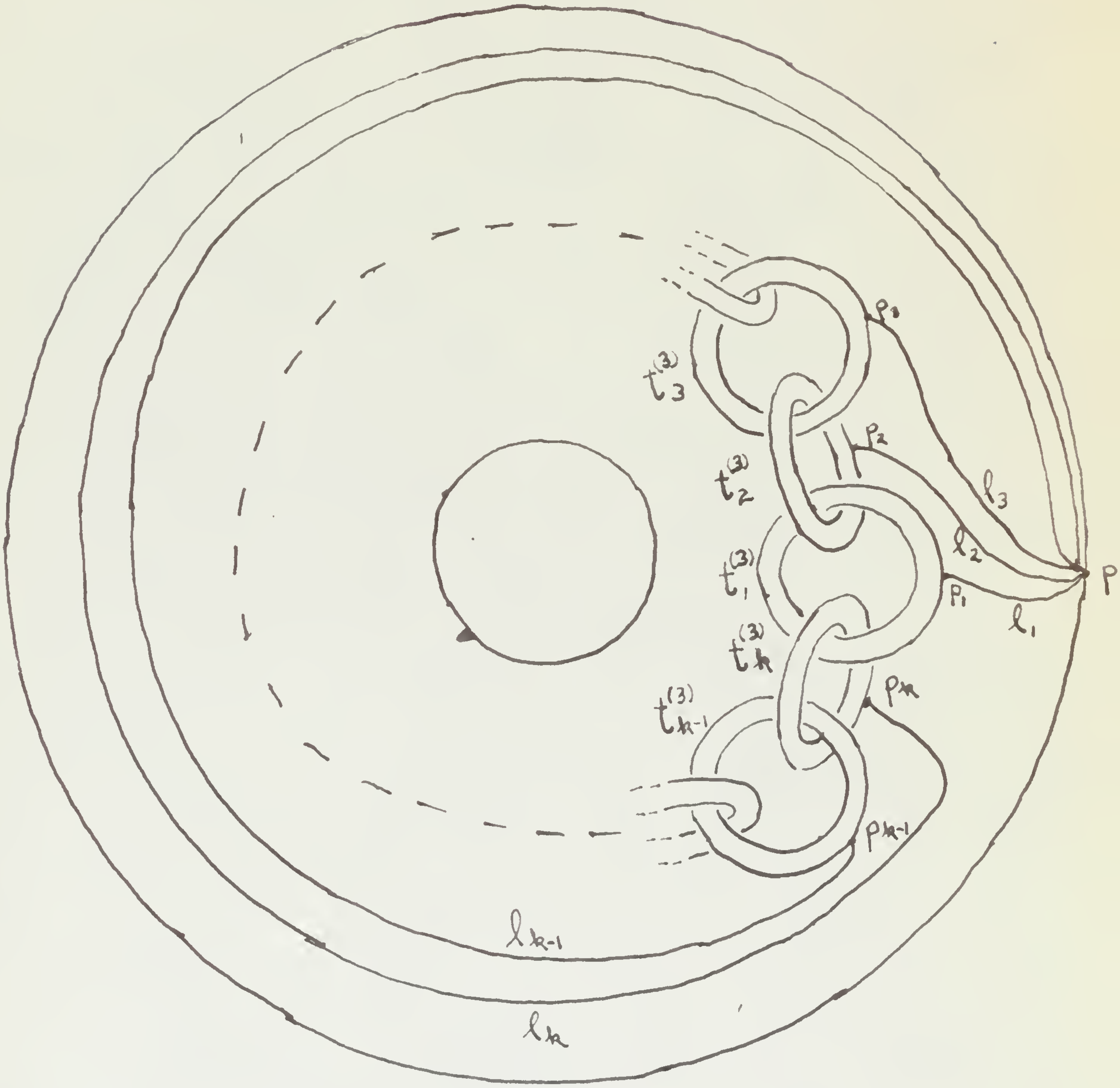
Let ℓ_i, i = 1, . . . k, be the simple arcs from p to p_i, indicated in figure 4. Then we have :

x is a loop on the surface of $t^{(3)}$, bounding in $S^3 - t^{(3)}$, and directed counter-clockwise in the diagram shown on next page.

y is a loop on the surface of $t^{(3)}$, bounding in $t^{(3)}$.

$a_i = \ell_i \bar{y}_i \ell_i^{-1}$, where $\bar{y}_i$ is defined by $\bar{y}_i = f_i(y)$, assuming that the orientation of the homeomorphism $g_i : t^{(2)} \simeq t_i^{(2)}$ has been chosen properly.

Since $\overline{T - A_1}$ is homeomorphic to the topological product of $\overline{t^{(3)} - \sigma}$ and a circle, it follows that $\pi_1(T - A_1)$ is the direct product of figure 4.



with an infinite cyclic group. We may take the path generated by rotating ρ about the (x_1, x_2) plane as a representative of the new generator, z, thus introduced.

We are now in a position to write down explicitly, as mappings of the interval $0 \leqslant s \leqslant 1$, the loops we have referred to. They are

$x(s) = (\rho, 2\pi s, 0)$, in T - coordinates
$z(s) = (\rho, 0, 2\pi s)$, in T - coordinates
$y(s) = (0, r_2 + r_1 \cos 2\pi s, r_3 + r_1 \sin 2\pi s, 0)$, Cartesian coordinates.

If we define

$$\begin{aligned} \bar{x}_i &= f_i(x) \\ \bar{y}_i &= f_i(y) \\ \bar{z}_i &= f_i(z) \\ \bar{a}_{ij} &= f_i(a_j) \end{aligned} ,$$

These are generators of $\pi_1(\overline{T_i - \bigcup_j T_{ij}})$, based at p_i. If we set

$$\begin{aligned} x_i &= l_i \bar{x}_i l_i^{-1} \\ y_i &= l_i \bar{y}_i l_i^{-1} \\ z_i &= l_i \bar{z}_i l_i^{-1} \\ a_{ij} &= l_i \bar{a}_{ij} l_i^{-1} \end{aligned}$$

then we can regard these as elements of $\pi_1(\overline{T - A_1})$, and the first three can also be regarded as elements of $\pi_1(\overline{T - A_0})$.

Referring to the definitions of the homeomorphisms f_i (p.9) and to the representations of x_i, z as mappings of the unit interval, we see that

$$\overline{x}_i(s) = (\rho_i, 0, 2\pi s)_i \qquad (T_i \text{ - coordinates})$$

$$\overline{z}_i(s) = (\rho_i, 2\pi s, 0)_i \qquad (T_i \text{ - coordinates})$$

Thus $\overline{x}_i$ is the loop generated by rotating ρ_i about the (x_1, x_2) plane, and $\overline{z}_i$ is the loop on the surface of $t_i^{(3)}$ and bounding in $S^3 - t_i^{(3)}$. Hence $x_i = l_i \overline{x}_i l_i^{-1}$ can be seen to be homotopic to z in $\overline{T - A_1}$, the homotopy being accomplished simply by moving the generating point, ρ_i , backwards along the arc l_i to ρ . We can also compute $z_i = l_i \overline{z}_i l_i^{-1}$ (considered as an element of $\pi_1(T - A_1)$) from figure 3. We obtain

$$\begin{aligned}
z_1 &= b_2 x^{-1} b_k^{-1} x &&= a_2 x^{-1} a_k^{-1} x \\
z_2 &= && a_3^{-1} a_1 \\
z_3 &= b_4 b_2^{-1} &&= a_4 a_2^{-1} \\
&\vdots \\
z_{k-1} &= b_k b_{k-2}^{-1} &&= a_k a_{k-2}^{-1} \\
z_k &= \alpha^{-1} a_{k-1} &&= x a_1^{-1} x^{-1} a_{k-1}
\end{aligned}$$

It is also obvious that

$$x_i = z$$

$$y_i = a_i$$

Since x_i, y_i, and z_i may be considered as elements of $\pi_1(l_i \cup \text{bdy}\, T_i)$, the above relations may be considered as defining the injection homomorphisms:

$$\Pi_1(l_i \cup \mathrm{bdy}\, T_i) \longrightarrow \Pi_1(\overline{T-A_1}) .$$

We are thus in a position to apply the usual procedure for computing $\Pi_1((\overline{T-A_1}) \cup (\overline{T_i - \bigcup_j T_{ij}}) \cup l_i)$. The group will be given by

Generators: $x, y, z, x_i, y_i, z_i, a_j, a_{ij} \quad ; \; j = 1, 2, \cdots k.$

Relations:

$$\Pi_1(\overline{T-A_1}) \begin{cases} [x,y] = [y,z] = [z,x] = [z,a_j] = 1 & ; \; j = 1,2,\cdots k \\ y = [a_j^{-1}, a_{j-1}] = [a_j^{-1}, a_{j+1}] & ; \; j = 2,4,\cdots\cdots k-2 \\ y = [a_k^{-1}, a_{k-1}] = [a_k^{-1}, a_1^{x}] & \end{cases}$$

$$\Pi_1((\overline{T_i - \bigcup_j T_{ij}}) \cup l_i) \begin{cases} [x_i, y_i] = [y_i, z_i] = [z_i, x_i] = [z_i, a_{ij}] & ; \; j = 1,2,\cdots k \\ y_i = [a_{ij}^{-1}, a_{ij-1}] = [a_{ij}^{-1}, a_{ij+1}] & ; \; j = 2,4,\cdots k-2 \\ y_i = [a_{ik}^{-1}, a_{ik-1}] = [a_{ik}^{-1}, a_{i1}^{x_i}] & \end{cases}$$

Injection Relations

$$\begin{cases} x_i = z \\ y_i = a_i \\ z_i = \begin{cases} a_{i+1}^{-1} a_{i-1} & \text{if } i \text{ is even and } i \neq k \\ a_{i+1} a_{i-1}^{-1} & \text{if } i \text{ is odd and } i \neq 1 \\ a_2 x^{-1} a_k^{-1} x & \text{if } i = 1 \\ x a_1^{-1} x^{-1} a_{k-1} & \text{if } i = k \end{cases} \end{cases}$$

Noting that all of the sets $(\overline{T_i - \bigcup_j T_{ij}}) \cup l_i$ are pairwise disjoint except at the base point p , we may adjoin them all simultaneously and obtain $\pi_1(\overline{T - A_2})$ simply by allowing i to range from 1 to k in the presentation shown on the preceding page.

By the obvious induction argument, we can now state that $\pi_1(\overline{T - A_r})$ is given by

Generators: $x_\alpha, y_\alpha, z_\alpha, a_{\alpha i} \quad ; i=1,\cdots k \quad , \quad l(\alpha) \leq r-1$

and

Relations:

$$[x_\alpha, y_\alpha] = [y_\alpha, z_\alpha] = [z_\alpha, x_\alpha] = [z_\alpha, a_{\alpha i}] \quad ; \; i=1,\cdots k \, , \; l(\alpha) \leq r-1$$

$$y_\alpha = [a^{-1}_{\alpha i}, a_{\alpha i-1}] = [a^{-1}_{\alpha i}, a_{\alpha i+1}] \quad ; \; i=2,4,\cdots k-2 \, , \; l(\alpha) \leq r-1$$

$$y_\alpha = [a^{-1}_{\alpha k}, a_{\alpha k-1}] = [a^{-1}_{\alpha k}, a_{\alpha 1}^{x_\alpha^{-1}}] \quad ; \quad l(\alpha) \leq r-1$$

$$x_{\alpha i} = z_\alpha \quad ; \; i=1,2,\cdots k \, , \; l(\alpha) \leq r-2$$

$$y_\alpha = a_\alpha \quad ; \quad l(\alpha) \leq r-1$$

$$z_{\alpha i} = \begin{cases} a^{-1}_{\alpha i+1} a_{\alpha i-1} & \text{if } i \text{ is even and} \neq k \, , \quad l(\alpha) \leq r-2 \\ a_{\alpha i+1} a^{-1}_{\alpha i-1} & \text{if } i \text{ is odd and } \neq 1 \, , \quad l(\alpha) \leq r-2 \\ a_{\alpha 2} x_\alpha^{-1} a^{-1}_{\alpha k} x_\alpha & \text{if } i = 1 \quad l(\alpha) \leq r-2 \\ x_\alpha a^{-1}_{\alpha 1} x_\alpha^{-1} a_{\alpha k-1} & \text{if } i = k \quad l(\alpha) \leq r-2 \end{cases}$$

The injections $\pi_1(\overline{T - A_r}) \longrightarrow \pi_1(\overline{T - A_{r+1}})$ assign to each element of the first group the same-named element of the second group. Hence, the group $\pi_1(T - A)$ is the limit of the direct homomorphism sequence $\{\pi_1(T-A_r)\}$[†], and is obtained merely by allowing α , in the presentation on preceding page to range over all arrays of finite length. The group $\pi_1(S^4 - A)$ is obtained by adjoining the two relations $x = z = 1$.

[†] See (5), footnotes 8 and 9.

To show that $\pi_1(S^4 - A)$ is non-trivial we exhibit the following representation:

$$x_\alpha = z_\alpha = 1 \qquad \text{for all } \alpha .$$

$$\begin{aligned} y &= (1,2)(3,4), \\ y_i &= (1,2)(5,6) \quad \text{if } i \text{ is odd}, \\ y_i &= (1,3)(2,4) \quad \text{if } i \text{ is even}; \end{aligned}$$

and, inductively,

$$\begin{aligned} \text{if} \quad & y_\alpha = (a,b)(c,d), \\ \text{set} \quad & y_{\alpha i} = (a,b)(e,f) \quad \text{if } i \text{ is odd}, \\ & y_{\alpha i} = (a,c)(b,d) \quad \text{if } i \text{ is even}, \end{aligned}$$

where e, f are the two integers between 1 and 6, not among a, b, c, d.

Computation of $\pi_1(S^n - A)$, $n \geq 4$.

The procedure here is almost identical with that for the case $n = 4$, so we shall only sketch it, filling in the significant departures from the previous case.

First we use the result from the previous calculation of the group $\pi_1(\overline{t^{(3)} - \sigma})$. For $\pi_1(\overline{T - A_1})$ we merely adjoin $(n-3)$ new generators which must lie in the center of $\pi_1(\overline{T - A_1})$. To make the notation uniform we replace x by $x^{(2)}$, z by $x^{(3)}$ and call the new generators $x^{(4)}, x^{(5)}, \ldots x^{(n-1)}$. Then $x^{(q)}$ is represented in T-coordinates by the loop

$$x^{(q)}(s) = (\rho, 0, 0, \cdots 0, 2\pi s, 0, \cdots 0), \quad 0 \leq s \leq 1,$$

where the $2\pi s$ occurs in the eq-coordinate position.

As before, we define

$$\bar{x}_i^{(q)} = f_i(x^{(q)}), \quad \bar{y}_i = f_i(y), \quad \bar{a}_{ij} = f_i(a_j),$$

and

$$x_i^{(q)} = l_i \bar{x}_i^{(q)} l_i^{-1}, \quad y_i = l_i \bar{y}_i l_i^{-1}, \quad a_{ij} = l_i \bar{a}_{ij} l_i^{-1}.$$

Then $x_i^{(q)}$ and y_i can be considered as generators of $\pi_1(l_i \cup \text{bdy } T_i)$, and we proceed to compute the injection relations. We have

$$\begin{aligned}
\bar{x}_i^{(2)} &= f_i(\rho, 2\pi s, 0, 0, \cdots) = (\rho_i, 0, 0, \cdots 2\pi s)_i \\
\bar{x}_i^{(3)} &= f_i(\rho, 0, 2\pi s, 0, 0, \cdots) = (\rho_i, 2\pi s, 0, 0, \cdots)_i \\
\bar{x}_i^{(4)} &= f_i(\rho, 0, 0, 2\pi s, 0, 0, \cdots) = (\rho_i, 0, 2\pi s, 0, \cdots)_i \\
\vdots \quad & \vdots \qquad\qquad\qquad\qquad \vdots \\
\bar{x}_i^{(n-1)} &= f_i(\rho, 0, 0, \cdots, 2\pi s) = (\rho_i, 0, 0, \cdots, 2\pi s, 0)_i
\end{aligned}$$

Hence, by the same reasoning as before, recalling that the higher T- and T_i - coordinates of a point are identical, we obtain

$$x_i^{(2)} = x^{(n-1)}$$

$$x_i^{(3)} = \begin{cases} a_{i+1}^{-1} a_{i-1} & \text{if } i \text{ is even} \neq k \\ a_{i+1} a_{i-1}^{-1} & \text{if } i \text{ is odd} \neq 1 \\ a_2 x^{(2)^{-1}} a_k^{-1} x^{(2)} & \text{if } i = 1 \\ x^{(2)} a_1^{-1} x^{(2)^{-1}} a_{k-1} & \text{if } i = k \end{cases}$$

$$x_i^{(4)} = x^{(3)}, \quad x_i^{(5)} = x^{(4)}, \cdots\cdots, \quad x_i^{(n-1)} = x^{(n-2)}, \quad y_i = a_i .$$

Thus, taking the generators $x^{(q)}, x_i^{(q)}, y, y_i, a_j, a_{ij}$ with the usual relations and the foregoing injection relations, we obtain $\pi_1\big((\overline{T-A_1}) \cup (\overline{T_i - \bigcup_j T_{ij}}) \cup \ell_i\big)$. As before, we can allow i to range from 1 to k, obtaining $\pi_1(\overline{T - A_2})$. From this we can readily infer that $\pi_1(\overline{T - A_r})$ is given by

Generators: $x_\alpha^{(q)}, y_\alpha, a_{\alpha i}$; $q = 2, 3, \cdots n-1$, $i = 1, \cdots k$, $\ell(\alpha) \leq r-1$

and

Relations

$$
\begin{array}{lll}
[x_\alpha^{(q)}, y_\alpha] = 1 & q = 2, \cdots n-1, & \ell(\alpha) \leq r-1 \\
[x_\alpha^{(q)}, a_{\alpha i}] = 1 & q = 3, \cdots n-1, & \ell(\alpha) \leq r-1 \\
y_\alpha = [a_{\alpha i}^{-1}, a_{\alpha i-1}] & i = 2, 4, \cdots k, & \ell(\alpha) \leq r-1 \\
\quad = [a_{\alpha i}^{-1}, a_{\alpha i+1}] & i = 2, 4, \cdots k-2, & \ell(\alpha) \leq r-1 \\
\quad = [a_{\alpha k}^{-1}, a_{\alpha 1}^{x_\alpha^{(2)-1}}] & , & \ell(\alpha) \leq r-1 \\
x_{\alpha i}^{(4)} = x_\alpha^{(3)}, & i = 1, 2, \cdots k, & \ell(\alpha) \leq r-2 \\
x_{\alpha i}^{(5)} = x_\alpha^{(4)} & i = 1, 2, \cdots k & \ell(\alpha) \leq r-2 \\
\vdots & & \\
x_{\alpha i}^{(n-1)} = x_\alpha^{(n-2)} & i = 1, 2, \cdots k & \ell(\alpha) \leq r-2 \\
x_{\alpha i}^{(2)} = x_\alpha^{(n-1)} & i = 1, 2, \cdots k & \ell(\alpha) \leq r-2
\end{array}
$$

$$
x_{\alpha i}^{(3)} = \left\{ \begin{array}{ll}
a_{\alpha i+1}^{-1} a_{\alpha i-1} & i = 2, 4, \cdots k-2 \\
a_{\alpha i+1} a_{\alpha i-1}^{-1} & i = 3, 5, 7, \cdots k-1 \\
a_{\alpha 2} x_\alpha^{(2)-1} a_{\alpha k}^{-1} x_\alpha^{(2)} & i = 1 \\
x_\alpha^{(2)} a_{\alpha 1}^{-1} x_\alpha^{(2)-1} a_{\alpha k-1} & i = k
\end{array} \right\} \quad \ell(\alpha) \leq r-2
$$

$$
y_\alpha = a_\alpha \qquad \ell(\alpha) \leq r-1
$$

Since the injections $\pi_1(\overline{T - A_r}) \rightarrow \pi_1(\overline{T - A_{r+1}})$ are defined by leaving the names of elements unchanged, we see that $\pi_1(T-A)$, being the direct limit of the sequence $\{\pi_1(\overline{T - A_r})\}$, is given by the above, where α is allowed to range over all arrays of finite length. $\pi_1(S^n - A)$ is of course obtained by adjoining the relations

$$x^{(q)} = 1, \quad q = 2, 3 \ldots\ldots n-1 .$$

A non-trivial representation of $\pi_1(S^n - A)$ in the symmetric group, S_6, is given by

$$x^{(q)}_{\alpha} = 1 \qquad q = 2, 3, \ldots . \; n-1, \quad \text{all } \alpha ,$$

$$y = (1,2)\ (3,4),$$

$$y_i = a_i = (1,2)\ (5,6) \quad \text{if } i \text{ is odd},$$

$$y_i = a_i = (1,3)\ (2,4) \quad \text{if } i \text{ is even};$$

and, inductively,

if $\quad y_{\alpha} = (a,b)\ (c,d),$

then set $a_{\alpha i} = y_{\alpha i} = (a,b)\ (e,f)$ if i is odd,

$\qquad a_{\alpha i} = y_{\alpha i} = (a,c)\ (b,d)$ if i is even,

where e, f are the two integers between 1 and 6 different from a, b, c, and d.

SECTION III

Since the complement of a compact zero-dimensional set in R^n can be multiply connected, one is naturally led to ask if such a set can exist in Hilbert space, R^ω. The answer is decidedly negative, as indicated in the following theorem:

Theorem III A. Let K be a compact set in Hilbert space, R^ω. Then $R^\omega - K$ is contractible.

First we prove

Lemma III A. If K is a compact subset of R^ω and

$$D_i = \max_{x \varepsilon K} (x_i),$$

then $\lim_{i \to \infty} D_i = 0$

Proof.

We note that compactness of K implies the existence of the D_i and existence of points

$$P_i = (P_{1i}, P_{2i}, \cdots P_{i-1,i}, D_i, P_{i+1,i}, \cdots\cdots)\varepsilon K.$$

Now suppose that infinitely many of the D_i are bounded from zero. Choose a convergent subsequence, $\{P_{\varphi(i)}\}$ for which the D_i are bounded from zero, and let $Q = (q_1, q_2, \cdots)$ be their limit point. Then

$$\rho^2(P_{\varphi(i)}, Q) = \sum_{j \neq \varphi(i)} (P_{j\varphi(i)} - q_j)^2 + (D_{\varphi(i)} - q_{\varphi(i)})^2 \longrightarrow 0$$

So we must have

$$D_{\varphi(i)} - q_{\varphi(i)} \longrightarrow 0$$

But $q_{\varphi(i)} \to 0$ since $q \varepsilon R^\omega$. Hence $D_{\varphi(i)} \to 0$, contradicting the assumption that $D_{\varphi(i)}$ were bounded from zero.

Proof of Theorem III A.

Let $D_i \to 0$ be as defined in lemma III A, and define $\delta_i = |D_i| + \frac{1}{i}$. Then the δ_i are all positive and tend to zero as $i \to \infty$.

For $x \in R^\omega$, $0 \leq t \leq 1$, define

$$f_t(x) = \begin{cases} x \text{ if } t = 0 \\ (x_1, x_2, \cdots x_{n-1}, \delta_n, \delta_{n+1}, 0, 0, \cdots) \text{ if } t = 1/n \\ \text{linear in } t \text{ for } \frac{1}{n+1} \leq t \leq \frac{1}{n} \end{cases}$$

For any positive value of t, say $\frac{1}{n+1} \leq t \leq \frac{1}{n}$, we note that the (n + 1)st coordinate of $f_t(x)$ is $\delta_{n+1} > D_{n+1}$, and hence $f_t(x)$ cannot lie in K. Further, since f_0 is the identity and $f_1(R^\omega) = (\delta_1, \delta_2, 0, 0, 0, \cdots\cdots)$, the function f_t is a contraction of $R^\omega - K$, provided we can assure continuity. Now f is clearly continuous in (t, x) for t > 0. For t = 0 we have

$$\begin{aligned} \rho(f_0(x), f_t(y)) &\leq \rho(f_0(x), f_0(y)) + \rho(f_0(y), f_t(y)) \\ &\leq \rho(x, y) + \rho(y, f_t(y)) \end{aligned}$$

For $\frac{1}{n+1} \leq t \leq \frac{1}{n}$, we have

$$\begin{aligned} y &= (y_1, y_2, \cdots\cdots y_{n-1}, \quad y_n \quad, y_{n+1}, \quad y_{n+2}, y_{n+3} \cdots) \\ f_t(y) &= (y_1, y_2, \cdots\cdots y_{n-1}, \varepsilon\delta_n + (1-\varepsilon) y_n, \delta_{n+1}, (1-\varepsilon)\delta_{n+2}, 0, 0 \cdots) \end{aligned}$$

where $0 \leq \varepsilon \leq 1$

Hence

$$\begin{aligned} \rho^2(y, f_t(y)) &= \varepsilon^2(\delta_n - y_n)^2 + (\delta_{n+1} - y_{n+1})^2 + (y_{n+2} + (\varepsilon - 1)\delta_{n+2})^2 + \sum_{n+3}^{\infty} y_i^2 \\ &\leq 4(\delta_n^2 + \delta_{n+1}^2 + \delta_{n+2}^2) + \sum_{n+3}^{\infty} y_i^2 \end{aligned}$$

Since δ_n, and $\sum_n^\infty y_i^2$ both tend to zero as $n \to \infty$, it is clear that we can make $\rho(y, f_t(y))$ arbitrarily small by choosing t small enough. This assures the continuity of f in (t,x) at $t=0$.

Corallary: If $C \subset K \subset R^\omega$, and K is compact, then $R^\omega - C$ is contractible.

Proof: We merely need to reexamine the contraction defined above, and note that $f_t(x)$ is defined for all $x \in R^\omega$ and $f_t(x)$ cannot lie in K if $t > 0$.

The conditions on C can be weakened much further without impairing the conclusion. For example, if C is bounded along infinitely many axes, with the bounds tending to zero, then $R^\omega - C$ is still contractible. Deeming this more or less irrelevant to the present paper, we shall not stop to consider these various refinements.

I have not been able to determine whether or not a set of finite dimension can leave the Hilbert cube, I_ω, multiply connected. However if such a set exists it is surely pathological.

For the purpose of investigating such sets in I_ω we define the sets

$$T_n(x) = \{ y \mid y_i = x_i \text{ if } i \geq n \; ; |y_i| \leq \tfrac{1}{i} \text{ if } i < n \} \quad \text{for } x \in I_\omega$$

and

$$T_n(A) = \bigcup_{x \in A} T_n(x) \qquad \text{for } A \subset I_\omega .$$

We note that for two sets A, B in I_ω, we have

$$A \cap T_n(B) = 0 \Leftrightarrow B \cap T_n(A) = 0$$

Consider the property $P(C)$ of a set C in I_ω defined by

$$P(C) :\equiv: x \in I_\omega - C \Rightarrow \exists n \ni T_n(x) \cap C \neq 0$$

or, equivalently,

$$P(C) :\equiv: I_\omega = \bigcup_{n=1}^{\infty} T_n(C)$$

If we now define $p_n(x)$ to be the projection,

$$p_n(x) = (0, 0, \cdots 0, x_n, x_{n+1}, \cdots\cdots),$$

then we immediately obtain:

Theorem III B. If the property $P(C)$ holds, then, for some n,

$$\dim p_n(C) = \infty .$$

Proof. If $P(C)$ holds we have

$$I_\omega = \bigcup_{n=1}^{\infty} T_n(C)$$

But I_ω cannot be a countable union of sets of finite dimension[/]. Hence, for some n,

$$\dim T_n(C) = \infty$$

But $T_n(C)$ is obviously the topological product of $p_n(C)$ with the n-1-dimensional cube I^{n-1}. Hence

$$\dim p_n(C) = \infty \quad .$$

[/] See (6), page 49 .

We now prove:

Theorem III C. If $P(C)$ does not hold, then $I_\omega - C$ is contractible.

Proof: We have

$$P(C) \text{ false} \Rightarrow \exists\, a \,\varepsilon\, I_\omega - C \ni. \text{ for all } n,\ T_n(a) \cap C = 0$$

If we define

$$R_n(x) = (x_1, x_2, \dots\dots x_{n-1}, a_n, a_{n+1}, a_{n+2}, \dots\dots)$$

Then

$$R_n(x) \,\varepsilon\, T_n(a)$$

Thus we can define a contraction of $I_\omega - C$ by

$$f_t(x) = \begin{cases} x & \text{if } t = 0 \\ R_n(x) & \text{if } t = \frac{1}{n} \\ & \text{linear on } \frac{1}{n+1} \leq t \leq \frac{1}{n} \end{cases}$$

Then we note that if $t > \frac{1}{n}$, then $f_t(x)$ $I_\omega - C$. Moreover, $f_0(x) =$ identity and $f_1(x) = a$. The continuity of $f_t(x)$ is easily verified by a method similar to that of Theorem III A.

Combining Theorems III B and III C we obtain

Theorem III D. If $C \subset I_\omega$ and for every n, the projection $p_n(C)$ is of finite dimension, then $I_\omega - C$ is contractible.

A theorem analogous to but not nearly as strong as Theorem III D holds for compact sets in R^n. This is stated in Theorem III E, the proof of which requires the following lemma:

Lemma III B. Let k be a compact zero-dimensional set in R^{n-1}, $n \geqslant 3$, and let ε be an arbitrary positive number. Then there exists a finite covering $\{U_i\}$ of k such that

1). U_i is open in R^{n-1}

2). diam $U_i < \varepsilon$

3). $k \cap \text{bdy}\, U_i = 0$

4). $\overline{U_i} \cap \overline{U_j} = 0$ if $i \neq j$

5). $\overline{U_i}$ is a finite polyhedron

6). bdy U_i is connected.

Proof. The existence of a covering $\{Q_i\}$ satisfying conditions 1). to 5). was demonstrated by Antoine(2), but I shall briefly sketch a proof here.

First cover k with a finite number of open sets $\{\hat{Q}_i\}$, of diameter $< \varepsilon/2$ and whose boundaries do not intersect k. We may assume this covering to be refined so that the sets are disjoint. Since $\hat{Q}_i \cap k$ is compact, we can, by regularity of the space, find open sets $\tilde{Q}_i \subset \hat{Q}_i$ whose closures are disjoint, whose diameters are $< \varepsilon/2$, whose boundaries do not intersect k, and which cover k.

Now cover the boundary of each $\tilde{Q}_i$ with a finite number of open (n-1)-dimensional cubes, whose closures do not intersect k or the closures of any of the other sets of the covering. Let Q_i be the set $\tilde{Q}_i$ together with the open cubes covering its boundary. Then $\{Q_i\}$ covers k and satisfies conditions 1)., 3)., 4)., and 5)., and in addition the cubes can be chosen so small that diam $Q_i < \varepsilon/2$.

Suppose that for certain i, bdy Q_i is not connected. Let $B_{1i}, B_{2i}, \cdots B_{ri}$ be the components of bdy Q_i and let U^*_{ji} be the union of the bounded components of $R^{n-1} - B_{ji}$. Since diam $B_{ji} < \varepsilon/2$, the (n-2) sphere at distance $\varepsilon/2$ from a point of B_{ji} does not intersect B_{ji}. Thus all the bounded components of $R^{n-1} - B_{ji}$ are interior to this sphere, so that

$$\text{diam } U^*_{ji} < \varepsilon \quad .$$

We further see that

$$Q_i \subset \bigcup_j U^*_{ji} \ .$$

For let q be a point of Q_i. Then it lies in a bounded component of $R^{n-1} - \text{bdy } Q_i$. Hence the radial projection of bdy Q_i onto an (n-2)-sphere about q is an essential mapping[†]. Since the B_{ji} are compact and disjoint, it follows that this mapping is essential on at least one of them. Hence q lies in a bounded component of some $R^{n-1} - B_{ji}$ or $q \varepsilon U^*_{ji}$.

Since bdy $U^*_{ji} = B_{ji} \subset$ bdy Q_i , we see that $\{U^*_{ji}\}$ forms a covering of k satisfying conditions 1), 2)., 3)., 5)., and 6). Moreover, since the B_{ji} are closed, disjoint, and connected, we see that if two sets of the covering $\{U^*_{ji}\}$ intersect, then one of them contains the other. Hence it is possible to pick a subcovering $\{U_i\}$ satisfying all the conditions of the lemma.

[†] We make use here of the fact that for a closed set $C \subset R^{n-1}$, $n \geq 3$, a point q lies in a bounded component of $R^{n-1} - C$ if and only if the radial projection of C upon an (n-2)-sphere about q is essential. This theorem is easily deduced, for example, from Theorem VI 10, of (6).

We are now in a position to prove

Theorem III E. Let K be a compact set in R^n, $n \geqslant 3$, and let k be its projection in R^{n-1}. If k is zero-dimensional, then $R^n - K$ is simply connected.

Proof: Let L be a loop in $R^n - K$. We may assume L is polygonal. Choose

$$\varepsilon < \rho(K, L),$$

and take a covering $\{U_i\}$ of k satisfying conditions 1). to 6). of lemma III B.

Let V_i be the infinite cylinder over U_i. Since $\overline{U_i}$ is a finite polyhedron and L is polygonal, $L \cap \overline{V_i}$ consists of a finite number of polygonal arcs with end points on bdy V_i. We will show that each of these arcs can be deformed in R^n-K, with end points fixed, into an arc lying on bdyV_i. Doing this for all the U_i, we obtain a loop, $\tilde{L}$, homotopic to L and not intersecting $\bigcup_i V_i$. $\tilde{L}$ can then be raised, until it lies completely above K, and then contracted in the obvious manner.

Hence to complete the proof we must consider a polygonal arc ℓ of $L \cap \overline{V_i}$, and show that it can be deformed in R^n -K, with fixed end points, to an arc lying on bdy V_i. Let a and b be the end points of ℓ. They lie, of course, on bdy V_i. Let U_i^a be the cross-section of V_i through the point a. For a point x of V_i we denote the projection of x in U_i^a by x^* and the projection of ℓ in U_i^a by ℓ^*. Let $\widehat{xx^*}$ be the straight line segment from x to x^*. We contend that

$$x \varepsilon \ell \Rightarrow \widehat{xx^*} \wedge K = 0 .$$

For if $\widehat{xx^*}$ intersected K at some point p, then since ℓ is connected, there would be some point q in ℓ with the same ordinate as p. This would imply that p and q both lie in the set U_i^p and hence have distance less than ε apart. But this contradicts the choice of $\varepsilon < \rho(K,L)$.

In view of this, it is then obvious that ℓ is homotopic to the arc $\ell^* \cdot \widehat{b^*b}$. Now let $\tilde{\ell}$ be an arc on bdy U_i^a connecting a to b^*. This arc exists by property 6) of the covering $\{U_i\}$. Let N be a spherical neighborhood about a of radius ε. Then U_i^a, and hence $\tilde{\ell}$ and ℓ^* are in N. Since $a \varepsilon L$ and $\rho(K,L) > \varepsilon$, it follows that $N \wedge K = 0$. Hence we can deform ℓ^* into $\tilde{\ell}$ in $N < R^n - K$, with fixed end points. We have, then that $\ell \simeq \tilde{\ell}\,\widehat{b^*b}$ with fixed end points and $\tilde{\ell}\,\widehat{b^*b}$ lies on bdy V_i. This completes the proof.

As a result of the construction of Antoine, J. W. Alexander(4) constructed 2-cells and 3-cells in R^3 whose complements were not simply connected. We are now able to prove the following generalization of Alexander's theorem:

Theorem III F. For each q, $1 \leq q \leq n$, there exists in R^n, $n \geq 3$ a set homeomorphic to a q-cell whose complement is not simply connected.

Proof: Let the notation be the same as in section I, and take $e = E_0$ to be an n-cell not interior to T and intersecting T on its surface in an (n-1)-cell, B. Construct, in T, k disjoint n-cells $e_1, \ldots\ldots e_k$, intersecting bdy T in (n-1)-cells $b_1 \ldots\ldots b_k$ contained in B, and such that $e_i \cap T_i = B_i =$ an (n-1) cell on the surface of T_i, and $e_i \cap T_j = 0$ if $i \neq j$. We must make further assumptions on the unpathological nature of the cells e, B, e_i, b_i, B_i, but to avoid confusion we will introduce these assumptions at the point in the proof where they are used.

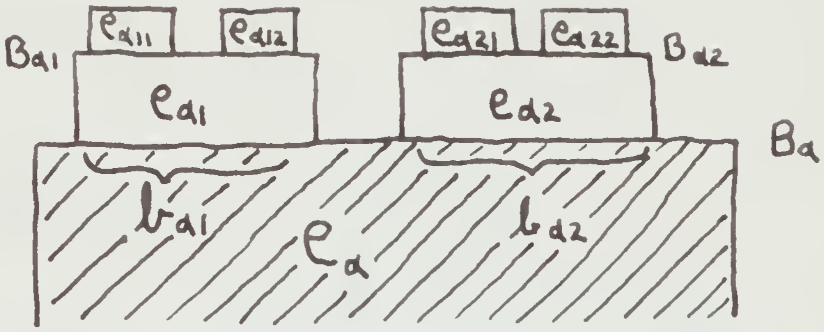
Now proceed in a similar manner in each of the tubes T_i, constructing disjoint n-cells e_{ij} from B_i to T_{ij}, such that

$$e_{ij} \cap T_{ij} = B_{ij} = \text{an (n-1)-cell}$$

$$e_{ij} \cap T_i = b_{ij} = \text{an (n-1)-cell} \subset B_i$$

$$e_{ij} \cap e_{\ell m} = 0 \quad \text{if } (i,j) \neq (\ell, m).$$

Continue the process ad infinitum, obtaining n-cells e_α, and (n-1)-cells b_α and B_α for all arrays α of finite length. We may represent the process schematically by the following diagram, taking k = n = 2.



We define the n-cells

$$E_1 = E_0 \cup \bigcup e_i$$
$$E_2 = E_1 \cup \bigcup e_{ij}$$
$$\vdots$$
$$E_r = E_{r-1} \cup \bigcup_{\ell(\alpha)=r} e_\alpha$$
$$\vdots$$

Also set

$$E_\infty = \bigcup_{r=0}^{\infty} E_r$$

and

$$E = \overline{E_\infty}$$

It is then obvious that

$$E - E_\infty = A \quad ,$$

where A is the zero-dimensional set constructed in section I. We further note that the n-cells e_α can be so chosen that representatives of the generators of $\pi_1(R^n - A)$ do not intersect E. Thus $\pi_1(R^n - E)$ will contain $\pi_1(R^n - A)$ as a subgroup and therefore be non-trivial.

We claim that E is an n-cell. To prove this we shall construct a homeomorphism, h, from E onto the n-cell

$$\widetilde{E} : -1 \leq x_i \leq 0 \qquad i = 1, 2, \cdots n.$$

Let F be the (n-1)-dimensional face of E defined by $x_n = 0$. Let $\widetilde{Q}_1, \widetilde{Q}_2, \ldots \widetilde{Q}_k$ be disjoint n-dimensional cubes in E which intersect F in (n-1)-cells, $F_1, F_2, \ldots F_k$ contained in F. Let $\widetilde{b}_1, \ldots, \widetilde{b}_k$ be the sets $\overline{\text{bdy } \widetilde{Q}_i - F_i}$, and let $\widetilde{E}_0$ be the n-cell $\overline{\widetilde{E} - \bigcup \widetilde{Q}_i}$. Then, if we have been careful to avoid pathological n- and (n-1)-cells, it is possible to construct a homeomorphism,

$$h_0 : E_0 \cong \widetilde{E}_0 ,$$

such that

$$h_0(b_i) = \widetilde{b}_i .$$

Next we repeat the process in each of the n-cells $\widetilde{Q}_i$, choosing disjoint n-cells $\widetilde{Q}_{i1}, \cdots, \widetilde{Q}_{ik}$ intersecting $\text{bdy } \widetilde{Q}_i$ in (n-1)-cells $F_{ij} \subset F_i$. Define

$$\widetilde{b}_{ij} = \overline{\text{bdy } \widetilde{Q}_i - F_{ij}}$$

$$\widetilde{e}_i = \overline{\widetilde{Q}_i - \bigcup_j \widetilde{Q}_{ij}}$$

$$\widetilde{E}_1 = \widetilde{E}_0 \cup \bigcup \widetilde{e}_i$$

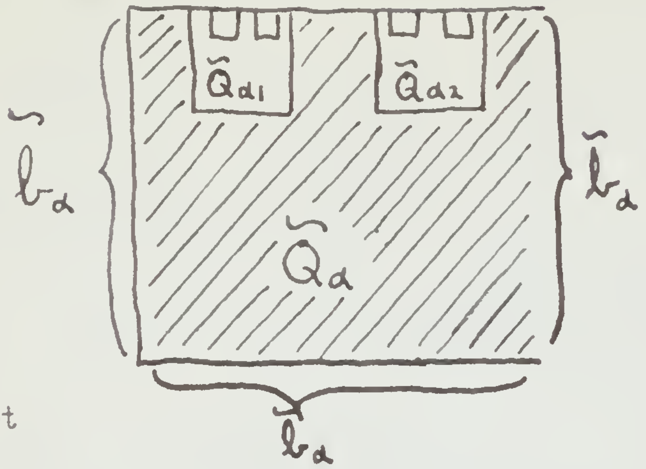
We can then construct a homeomorphism

$$h_1 : E_1 \cong \widetilde{E}_1$$

which is an extension of h_0 and such that

$$h_1(b_{ij}) = \widetilde{b}_{ij} .$$

Continuing this process ad infinitum, the general step of which we may represent by the diagram:



(shaded portion $= \tilde{e}_\alpha$),

we set

$$\tilde{E}_r = \tilde{E}_{r-1} \cup \bigcup_{l(\alpha)=r} \tilde{e}_\alpha$$

$$E_\infty = \bigcup_{r=0}^{\infty} E_r \quad .$$

This process constructs for us a homeomorphism

$$h_\infty : E_\infty \cong \tilde{E}_\infty \quad .$$

If we set $\tilde{A} = \tilde{E} - \tilde{E}_\infty$, then we see that, if we are careful that diam $\tilde{Q}_\alpha \to 0$ as $l(\alpha) \to \infty$, every point of A has a unique representation, e_β where β is an infinite array of the integers 1, 2, ... k , and

$$\tilde{e}_\beta = \bigcap_{m=1}^{\infty} \tilde{Q}_{\beta_m} = \lim_{m \to \infty} \tilde{e}_{\beta_m} \quad ,$$

where β_m is the finite array consisting of the first m integers of β . Furthermore the points of A have a similar unique representation,

$$e_\beta = \bigcap_{m=1}^{\infty} T_{\beta_m} = \lim_{m \to \infty} e_{\beta_m} \quad .$$

Finally, if we define $h : E \to \tilde{E}$ by

$$h = h_\infty \text{ on } E_\infty$$

and $h(e_\beta) = \tilde{e}_\beta$,

then it is relatively trivial to verify that h is a homeomorphism.

We have thus proved the theorem for the case $q = n$. For $1 \leq q \leq n$, we note that the sets $\{\tilde{Q}_\alpha\}$ can be chosen in such a way that $\tilde{A}$ lies on the x_1-axis. Thus, the map h^{-1} on the q-cell defined by $x_n = x_{n-1} = \cdots\cdots = x_{q+1} = 0$ picks out a q-cell which contains A and is contained in E and consequently has a complement which is not simply connected.

BIBLIOGRAPHY

1. Borsuk. Problème.
Colloquium Mathematicum Vol. I_3 (1948), p. 242.

2. Antoine, L. Sur L'homéomorphisme de Deux Figures et de Leurs Voisinages. Journal des Mathématiques Pures et Appliquées. 4 (1921) pp. 221-325.

3. Coelho, R. P. On the Groups of Certain Linkages.
Portugaliae Math. 6 (1947) pp. 57-65.

4. Alexander, J. W. Remark on a Construction of Antoine.
Bull. Acad. Sci. 1924.

5. Fox, R. H. and Artin, E. Some Wild Cells and Spheres in Three-dimensional Space. Annals of Mathematics, Vol. 49 No. 4 (1948) pp. 979-989.

6. Hurewicz and Wallman. Dimension Theory.
Princeton University Press (1941).

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